NUMBER THEORY AND DIOPHANTINE ANALYSIS.

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In the recurring series $u_0=3$, $u_1=0$, $u_2=2$, ..., where the scale of relation is $u_{n+3}=n_{n+1}+u_n$, prove that u_p is always divisible by p when p is prime. Is the converse true?

Solution by DR. L. E. DICKSON, The University of Chicago.

In this problem, we have $u_k=S_k$ for every k, where S_k denotes the sum of the kth powers of the roots of $x^3-x-1=0$. By the formula of Girard (often attributed to Waring),

(1) $$S_n=n\,\Sigma\,\frac{(m+l-1)!}{m!\,l!}\quad\left(\begin{matrix}\text{summed for all integers} \geqq 0\\ \text{for which } 2m+3l=n.\end{matrix}\right)$$

Hence, for n a prime number, $u_n=S_n$ is divisible by n.

The problem admits of a wide generalization. Given any integers m, p_1, ..., p_m, with m positive, the recursion formula

(2) $$z_{x+m}+p_1z_{x+m-1}+p_2z_{x+m-2}+\ldots+p_mz_x=0$$

has the solution* $z_y=\sum\limits_{i=1}^{m}C_ia_i^x$, in which the C's are arbitrary, while a_1, ..., a_m are the roots of

(3) $$a^m+p_1a^{m-1}+p_2a^{m-2}+\ldots+p_m=0.$$

The C_i may be expressed in terms of z_0, z_1, ..., z_{m-1}, and conversely. For suitably chosen (integral) values of the latter, we may make $C_1=1$, ..., $C_m=1$. Hence when relation (2) is arbitrarily assigned, we may construct an infinite series of integers z_0, z_1, ..., for which the recursion formula is (2) and such that

(4) $$z_k=S_k=\text{sum of } k\text{th powers of roots of (3)}.$$

Then z_p is given by Girard's formula

$$S_p=p\,\Sigma\,\frac{(-1)^{\lambda_1+\ldots+\lambda_m}(\lambda_1+\ldots+\lambda_m-1)!}{\lambda_1!\,\lambda_2!\,\ldots\,\lambda_m!}p_1^{\lambda_1}\ldots p_m^{\lambda_m},$$

summed for all integers $\lambda_i\geqq 0$ for which $\lambda_1+2\lambda_2+\ldots+m\lambda_m=p$. *Hence, for* $p_1=0$, *and* p *a prime*, z_p is divisible by p.

The only solution of the a's are distinct (*Encyclopaedie Mathematik*, Vol. 1, p. 934).