# SOME SEQUENCES LIKE FIBONACCI'S

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## INTRODUCTION

Define a sequence $(T_n)$ of integers by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} + 1 \quad \text{when } n \text{ is even,}$$

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} - 1 \quad \text{when } n \text{ is odd,}$$

or, more concisely, by

(1) $$T_n = T_{n-1} + T_{n-2} + T_{n-3} + (-1) \ ,$$

with initial values

(2) $$T_1 = 0,\ T_2 = 2,\ T_3 = 3.$$

One of us (L.G.W.), playing with this sequence, had observed a number of apparent regularities, of which the most striking was that all positive prime numbers $p$ divide $T_p$—at least as far as hand computation was practicable. He then communicated his observations to the other of us, who—being a professional mathematician—did not know the reason for this phenomenon, but knew whom to ask. Light was shed on the properties of the sequence by D. H. Lehmer,* who proved that, indeed, $T_p$ is divisible by $p$ whenever $p$ is a positive prime number, and also confirmed the other observations made by one of us by experiment on some 200 terms of the sequence. [These further properties will not be referred to in the sequel—the reader, however, may wish to play with the sequence.]

In this note we shall present Lehmer's proof and state a conjecture of his, and then look at some other sequences with the same property.

## LEHMER'S PROOF

It is convenient to replace the definition (1) of our sequence $(T_n)$ by one that does not involve the parity of the suffix $n$, namely

(3) $$T_n = 2T_{n-2} + 2T_{n-3} + T_{n-4}.$$

This is arrived at by substituting

$$T_{n-1} = T_{n-2} + T_{n-3} + T_{n-4} + (-1)^{n-1}$$

in (1) and observing that $(-1)^{n-1} + (-1)^n = 0$. As the recurrence relation (3) is of order 4, we now need 4 initial values, say

(4) $$T_0 = 2,\ T_1 = 0,\ T_2 = 2,\ T_3 = 3.$$

It is well known that the general term of the sequence defined by (3) is of the form

(5) $$T_n = A\alpha^n + B\beta^n + C\gamma^n + D\delta^n,$$

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*The authors are greatly indebted, and deeply grateful, to Professor Lehmer for elucidating the properties of this sequence.

where $\alpha$, $\beta$, $\gamma$, $\delta$ are the roots of the "characteristic equation" of (3),

(6) $$f(x) \equiv x^4 - 2x^2 - 2x - 1 = 0,$$

and where the constants $A$, $B$, $C$, $D$ are determined from the initial values (4)

Put

$$S_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

so that the sequence $(S_n)$ satisfies the same recurrence relation as $(T_n)$. If $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$ are the elementary symmetric functions of the roots of (6), that is

$$\sigma_1 = \alpha + \beta + \gamma + \delta = 0,$$
$$\sigma_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -2,$$
$$\sigma_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = +2,$$
$$\sigma_4 = \alpha\beta\gamma\delta = -1,$$

where the values are read off the identity

$$f(x) \equiv x^4 - \sigma_1 x^3 + \sigma_2 x^2 - \sigma_3 x + \sigma_4,$$

then

$$S_1 = \sigma_1 = 0,$$
$$S_2 = \sigma_1^2 - 2\sigma_2 = 4,$$
$$S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 = 6,$$

and, of course,

$$S_0 = \alpha^0 + \beta^0 + \gamma^0 + \delta^0 = 4.$$

Thus, the initial values of $(S_n)$ are just twice those of $(T_n)$—see (4)—and it follows that

$$T_n = \frac{1}{2}S_n$$

for all $n$, or, equivalently, that

$$A = B = C = D = \frac{1}{2}$$

in (5).

We now use the formula

(7) $$(x + y + z + y)^p = x^p + y^p + z^p + t^p + p \cdot F_p(x,y,z,t),$$

where $p$ is a prime number, $x$, $y$, $z$, $t$ are arbitrary integers, and $F_p(x,y,z,y)$ is an integer that depends on them and on $p$. This identity stems from the fact that in the multinomial expansion of the left-hand side of (7), each term is of the form

$$\frac{p!}{i!j!k!l!}x^i y^j z^k t^l$$

with $i + j + k + l = p$; and the coefficient $\frac{p!}{i!j!k!l!}$ is divisible by $p$ unless one of the $i$, $j$, $k$, $l$ equals $p$ and the other three are zero. In our case, putting

$$x = \alpha,\ y = \beta,\ z = \gamma,\ t = \delta,$$

and recalling that $\alpha + \beta + \gamma + \delta = S_1 = 0$, we see that

$$S_p = -p \cdot F_p(\alpha, \beta, \gamma, \delta),$$

which is divisible by $p$. Thus also, $T_p = \frac{1}{2}S_p$ is divisible by $p$ when $p$ is an odd prime. But for $p = 2$ we also have this divisibility, as $T_2 = 2$. Thus, the following result is proved.

*Theorem 1:* If $p$ is a positive prime number, then $T_p$, defined by the recurrence relation (3) with initial values (4), is divisible by $p$.

D. H. Lehmer calls a composite number $q$ a *pseudoprime* for the sequence $(T_n)$ if $q$ divides $T_q$, and he conjectures that there are infinitely many such pseudoprimes. The smallest such pseudoprime is 30, and we have found no other. It may be remarked that when $q$ is a power of a prime number, say $q = p^d$, then $T_q$ is divisible by $p$ but not, as far as we have been able to check, by any higher power of $p$.

## OTHER SEQUENCES

Lehmer's argument presented above gives us immediately a prescription for making sequences of numbers, say $(U_n)$, defined by a linear recurrence relation and with the property that for prime numbers $p$ the $p$th term is divisible by $p$. All we have to ensure is that the roots of the characteristic equation add up to zero, and that the initial values give the sequence the right start. Thus, we have the following theorem.

*Theorem 2:* Let the sequence $(U_n)$ of numbers be defined by the linear recurrence relation of degree $d > 1$:

(8) $$U_n = a_2U_{n-2} + a_3U_{n-3} + \cdots + a_dU_{n-d}$$

with integer coefficients $a_2, a_3, \ldots, a$ and initial values

(9) $$U_0 = d,\ U_1 = 0,\ U_2 = 2a_2,\ U_3 = 3a_3,\ \ldots,$$

and, generally,

(10) $$U_i = \alpha_1^i + \alpha_2^i + \cdots + \alpha_d^i,$$

where $\alpha_1, \alpha_2, \ldots, \alpha$ are the roots of the characteristic equation

$$x^d - a_2x^{d-2} - a_3x^{d-3} - \cdots - a_d = 0,$$

and $i = 0, 1, 2, \ldots, d - 1$. Then $U_p$ is divisible by $p$ for every positive prime number $p$.

The proof is the same, *mutatis mutandis*, as that of Theorem 1, and we omit it here.

We remark that $d = 2$ is uninteresting: we get $U_{2m} = 2a_2^m$ when $n = 2m$ is even, and $U_n = 0$ when $n$ is odd. Thus, the first sequences of interest occur when $d = 3$. We briefly mention some examples.

*Example 1:* Put $d = 3$, $a_2 = 2$, $a_3 = 1$. The sequence can be defined by

$$U_n = U_{n-1} + U_{n-2} + (-1)^n,$$

which has the same growth rate, for $n \to \infty$, as the Fibonacci sequence. The pseudoprimes of this sequence, that is to say the positive composite integers $q$ that divide $U_q$, appear to include the powers 4, 8, 16, ... of 2.

*Example 2:* Put $d = 3$, $a_2 = 1$, $a_3 = 1$. The sequence becomes

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, \ldots,$$

with a much slower rate, for $n \to \infty$, than the Fibonacci sequence. The roots, say $\alpha$, $\beta$, $\gamma$, of the characteristic equation are approximately

$$\alpha = 1.324718,$$
$$\beta = -0.662359 + i \cdot 0.5622795,$$
$$\gamma = -0.662359 - i \cdot 0.5622795,$$

and as $n \to \infty$, the ratio of successive terms of our sequence tends to $\alpha$. This is substantially less than the ratio $\frac{1}{2} + \frac{1}{2}\sqrt{5} = 1.61803\ldots$ to which successive terms of the Fibonacci sequence tend. We have found no pseudoprimes for this sequence.

If the "dominant" root of the characteristic equation, that is the root with the greatest absolute value, is not single, real, and positive (if it is not real, then there is in fact a pair of dominant roots; and also in other cases there may be several dominant roots or repeated dominant roots), the sequence may oscillate between positive and negative terms, as it will also, in general, if continued backward to negative $n$.

*Example 3:* The sequence defined by

$$U_n = 3U_{n-2} - 2U_{n-3}$$

with initial values

$$U_0 = 3,\ U_1 = 0,\ U_2 = 6$$

has the property that positive prime numbers $p$ divide $U_p$. It can also be described, explicitly, by

$$U_n = (-2)^n + 2.$$

For positive $n$, from $n = 2$ on, the terms are alternatingly positive and negative.

These sequences have, like the Fibonacci sequence, suggested to one of the authors an investigation of certain groups, but this is not the place to describe the problems and results. They are related to those of Johnston, Wamsley, and Wright [1].

## REFERENCE

1. D. L. Johnston, J. W. Wamsley, & D. Wright, "The Fibonacci Groups," *Proc. London Math. Soc.* (3), 29 (1974):577-592.

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